

Strong Equivalence Relations for Iterated Models

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Abstract

The Iterated Immediate Snapshot model (IIS), due to its elegant geometrical representation, has become standard for applying topological reasoning to distributed computing. Its modular structure makes it easier to analyze than the more realistic (non-iterated) read-write Atomic-Snapshot memory model (AS). It is known that AS and IIS are equivalent with respect to *wait-free task* computability: a distributed task is solvable in AS if and only if it solvable in IIS. We observe, however, that this equivalence is not sufficient in order to explore solvability of tasks in *sub-models* of AS (i.e. proper subsets of its runs) or computability of *long-lived* objects, and a stronger equivalence relation is needed.

In this paper, we consider *adversarial* sub-models of AS and IIS specified by the sets of processes that can be *correct* in a model run. We show that AS and IIS are equivalent in a strong way: a (possibly long-lived) object is implementable in AS under a given adversary if and only if it is implementable in IIS under the same adversary. Therefore, the computability of any object in shared memory under an adversarial AS scheduler can be equivalently investigated in IIS.

1 Introduction

Iterated memory models (see a survey in [24]) proved to be a convenient tool to investigate and understand distributed computing. In an iterated model, processes pass through a series of disjoint communication-closed memories M_1, M_2, \dots . The most popular one is the *Iterated Immediate Snapshot* model (IIS) [4]. Processes access the memories one by one, each time invoking the *immediate snapshot* operation [3] that writes to the memory and returns a snapshot of the memory contents. Once memory M_k is accessed, a process never comes back to it. IIS has many advantages over the more realistic (non-iterated) read-write Atomic-Snapshot memory model (AS) [1]. Its modular structure makes it considerably easier to analyze algorithms and prove their correctness. Moreover, its nice geometrical representation [20, 23] makes it suitable for topological reasoning. It is natural therefore to seek for a generic transformation that would map any problem in AS to an *equivalent* problem in IIS.

It has been shown by Borowski and Gafni [4] that the complete sets of runs of IIS and AS are, in a strict sense, *equivalent*: a distributed task is (wait-free) solvable in AS if and only if it is (wait-free) solvable in IIS. They established the result by presenting a *forward simulation* that, in every AS run, simulates an IIS run [3], and a *backward simulation* that, in every IIS run, simulates an AS

run [4]. The equivalence turned out to be instrumental, e.g., in deriving the impossibility of wait-free set agreement [2, 19]. More generally, the equivalence enables the topological characterization of task solvability in AS [19, 15].

However, in order to investigate computability of *long-lived* objects or solvability of tasks in *sub-models* of AS (i.e., proper subsets of its runs), this equivalence turns out to be insufficient. The goal of this paper is to establish a stronger one using elaborate model simulations.

We focus on *adversarial* sub-models of AS [5, 21], specified by sets of processes that can be *correct* in a model run. Note that the original AS model is described by the adversary consisting of *all* non-empty sets of processes. Since the introduction of adversaries in [5], the models have become popular for investigating task computability [10, 11, 17]. But how to define an IIS “equivalent” for an adversarial AS sub-model?

In IIS, a correct yet “slow” process may be never noticed by other processes: a process may go through infinitely many memories M_1, M_2, \dots without appearing in the snapshots of any other process. Instead, we specify adversarial sub-models of IIS using the sets of *strongly correct* processes [26] (sometimes also referred to as *fast* processes [12]). Informally, a process is strongly correct in an IIS run if it belongs to the largest set of processes that “see” each other infinitely often in the run. A topological characterization of task computability in sub-IIS models has been recently derived [12]: given a task T and an IIS sub-model M , topological conditions for solving T in M are provided. Is this characterization relevant for sub-AS models also or is it specific to IIS?

In this paper, we show that the answer is “yes”. We show that sub-models of IIS and AS that are governed by the same adversary are equivalent in a strong sense: An object is implementable in AS under a given adversary if and only if it is implementable in IIS under the same adversary. This holds regardless of whether the object is one-shot, like a distributed task, or long-lived, like a queue or a counter. To achieve this result, we present a two-way simulation protocol that provides an equivalent sub-IIS model for any sub-AS model and which guarantees that the set of correct processes in an AS run coincides with the set of strongly correct processes in the simulated IIS run, and vice versa:

- We propose an “AS to IIS” simulation which ensures that a correct (in AS) process is “noticed” infinitely often by other correct (in AS) processes in the simulated IIS run, even if the process is much slower than the others. To this goal, we simulate IIS steps with the RAP (Resolver Agreement Protocol) [11] and employ a “fair” simulation strategy—at each point, we first try to promote the most “left behind” process in the currently simulated run. Even if the RAP-based simulation “blocks” because of a disagreement between the simulators (unavoidable in asynchronous fault-prone systems [6]), we guarantee that the blocked process is eventually noticed by more advanced simulated processes.
- To obtain our “IIS to AS” simulation, we extend the multiple-shot IS simulations [3] with a “helping” mechanism, reminiscent to the one employed in the atomic-snapshot simulation [1]. Here even if a process i is not able to complete its simulated read, it may adopt the snapshot published by a concurrent process j , under the condition that j has seen the most recent write of i . Since every move by a strongly correct process is eventually seen by every other strongly correct process, we derive the desired property that every strongly correct process makes progress in the simulated run.

Equating the set of correct processes with the set of strongly correct processes in the iterated simulated run is illuminating, because our algorithms provide an iterated equivalent to any adver-

serial model [5, 21]. Our simulations also preserve the set of processes considered to be participating in the original run, which motivates the recent topological characterization of task computability in sub-IIS models [12].

An important property of our simulation algorithms is that they are model-independent, i.e., they deliver the promised guarantees without making any assumptions on the model runs. In this sense, the algorithms are *wait-free*.

Roadmap. Section 2 relates our results to earlier work. Our model definitions, including the discussion of the AS and IIS models, the definition of strongly connected processes in IIS, and the definition of a simulation, are given in Section 3. Sections 4 and 5 present our two-way simulation.

2 Related work

The IIS model introduced by Borowsky and Gafni [4] has become standard in topological reasoning about distributed computing [18, 2, 4, 19, 15]. The IIS model is precisely captured by the standard chromatic subdivision of the input complex [23, 20], and thus enables intuitive and elegant reasoning about its computability power, in particular, distinguishing solvable and unsolvable. The IIS model is equivalent to the classical read-write model with respect to (wait-free) task solvability [3, 4, 13, 26].

On the one hand, Borowsky and Gafni [3] have shown that one round of IIS can be implemented wait-free in AS, thus establishing a wait-free simulation of multi-round IIS. But the simulation only ensures that *one* correct process appears as strongly correct in the IIS run. Our algorithm ensures that *every* correct processes appear as strongly correct in the simulation.

On the other hand, IIS can simulate AS in the non-blocking manner, i.e., making sure that at least one process that participates in infinitely many rounds of IIS manages to simulate infinitely many steps of AS [4]. Later, Gafni and Rajsbaum [13] generalized the simulation of [4] to \mathcal{L} -resilient adversaries [11]. It guarantees that at least one set in \mathcal{L} will appear correct in the simulated execution. Raynal and Stainer [26] presented an extension of the simulation in [4] and sketched a proof that the extension simulates a run in which each the set of correct process in the simulated AS run is equal to the set of strongly correct processes in the “simulating” IIS run. In this paper, we propose an algorithm that achieves this property using the idea of the original atomic-snapshot implementation by Afek et al. [1], which we believe to be more intuitive and simpler to understand.

The relations between different simulation protocols are summarized in the following table:

	$\text{correct(AS)} \subseteq \text{str-correct(IIS)}?$	$\text{str-correct(IIS)} \subseteq \text{correct(AS)}?$
From AS to IIS		
Borowsky and Gafni [3]	$\exists p \in \text{correct(AS)} : p \in \text{str-correct(IIS)}$	✓
This paper	✓	✓
From IIS to AS		
Borowsky and Gafni [4]	✓	$\exists p \in \text{str-correct(IIS)} : p \in \text{correct(AS)}$
Gafni and Rajsbaum [13]	✓	$\exists X \subseteq \text{str-correct(IIS)} : X \subseteq \text{correct(AS)}$
Raynal and Stainer [26]	✓	✓
This paper	✓	✓

An informal definition of a strongly correct process in IIS was proposed by Gafni in [7] and formally stated by Raynal and Stainer in [26, 27]. The equivalence between adversarial restrictions of AS and IIS we establish in this paper motivated formulating a generalized topological

characterization of task computability in sub-IIS [12].

Our AS-to-IIS simulation presented in Section 4 offers a novel use of the Resolver Agreement Protocol (RAP) proposed in [11], where a set of simulators try to maintain the balance between the simulated processes by promoting the “most behind” process that is not “blocked.” Our IIS-to-AS simulation presented in Section 5 is based on the non-blocking simulation of [4], with the helping mechanism similar to the one used in the original atomic snapshot construction [1].

Herlihy and Rajsbaum [16] considered the problem of simulating task solutions in a variety of models, but their results only concern colorless tasks, which boils down to a very restricted notion of simulation. Rajsbaum et al. [25] introduced the Iterated *Restricted* Immediate Snapshot (*IRIS*) framework, where the restriction is defined via a specific failure detector on the per-round basis.

3 Definitions

In this section, we recall how the standard read-write and IIS models are defined, discuss the notion of a strongly correct process in the IIS model, and explain what we mean by simulating one model in another.

Standard shared-memory model. We consider a standard atomic-snapshot model (*AS*) in which a collection $\Pi = \{1, \dots, n\}$ of processes communicate via atomically updating their distinct registers in the memory and taking atomic snapshots of the memory contents. AS is equivalent to the standard read-write shared-memory model [1]. Without loss of generality, we assume that every process writes its input value in the first step and then alternates taking snapshots with updating its register with the result of its latest snapshot. This is known as a *full-information* protocol. We say that a process *participates* in a run E if it performs at least one update operation. Let $part(E)$ denote the set of participating processes in E . A process i is *correct* (or *live*) in E if i takes infinitely many steps in E . Let $correct(E)$ denote the set of live processes in E .

IIS model and strongly correct processes. In the IIS memory model, each process is supposed to go through a series of independent memories M_1, M_2, \dots . Each memory is accessed by a process with a single *immediate snapshot* operation [3].

A run E in IIS is a sequence of non-empty sets of processes $S_1 \supseteq S_2 \supseteq \dots$, with each $S_r \subseteq \{1, \dots, n\}$ consisting of those processes that participate in the r th iteration of immediate snapshot (IS). Furthermore, each S_k is equipped with an ordered partition: $S_r = S_r^1 \cup \dots \cup S_r^{n_r}$ (for some $n_r \leq n$), corresponding to the order in which processes are invoked in the respective IS.

Fix a run $E = S_1, S_2, \dots$. The processes $i \in S_1$ are called *participating*. If j appears in all the sets S_k , we say that j is *infinitely participating* in E . The sets of participating and infinitely participating processes in a run E are denoted $part(E)$ and $\infty-part(E)$, respectively.

If $i \in S_r$ (i *participates in round r*), let V_{ir} denote the set of processes appearing in i ’s r -th *snapshot* in E , defined as the union of all sets in the partition of S_r preceding and including $S_r^m \subseteq S_r$ such that $i \in S_r^m$: $V_{ir} = S_r^1 \cup \dots \cup S_r^m$. It is immediate that for all processes i, j and rounds r , such that i and j participate in r , the following properties are satisfied [3]: (self-inclusion) $i \in V_{ir}$; (containment) $V_{ir} \subseteq V_{jr} \vee V_{jr} \subseteq V_{ir}$; and (immediacy) $i \in V_{jr} \Rightarrow V_{ir} \subseteq V_{jr}$.

Our definitions can be interpreted operationally as follows. S_r is the set of processes accessing memory M_r , and each S_r^j is the set of processes obtaining the same *snapshot* after accessing M_r . Recall that in IS, the view of a process $i \in S_r^j$ is defined by the values written by the processes in $S_r^1 \cup \dots \cup S_r^j$.

It is convenient then to define, for each round r of E , a directed graph G_E^r with processes that participate in r as nodes and a directed edge from i to j if $j \in V_{ir}$. $G_E^{(r)}$ is the union of the graphs G_r, G_{r+1}, \dots .

We say that *process i is aware of round r of process j* in an IIS execution E if there exists a path from i to j in $G_E^{(r)}$.

The *participating set of a process i in a run E* , denoted by $\text{part}(E, i)$ (or maybe $\text{aware}(i, E)$), is the set of processes that i is aware of their first round.

A process is *strongly correct* (or *fast* [12]) in E if in every round by every process in $\infty\text{-part}(E)$. Let $\text{str-correct}(E)$ denote the set of strongly correct processes in E . Intuitively, $\text{str-correct}(E)$ is the largest set of processes that “see” each other (appear in each other’s views) infinitely often in E . Formally, denote by G_E^* the graph limit $\lim_{r \rightarrow \infty} G_E^{(r)}$. That is, i is a vertex of G_E^* if it is in $\infty\text{-part}(E)$ and (i, j) is an edge of G_E^* if E contains infinitely many rounds r such that $j \in V_{ir}$, i.e., i is aware of infinitely many rounds of j .

Let $SC(E)$ be the set of processes in the strongly connected component of G_E^* . By the containment property of IIS snapshots, in every round r , either $i \in V_{jr}$ or $j \in V_{ir}$. Hence, for all $i, j \in \infty\text{-part}(E)$, we are guaranteed that G_E^* contains at least one of the edges (i, j) and (j, i) . Therefore, G_E^* has a single sink. In the following, we use some properties of strongly correct processes:

Proposition 1 *For all E in IIS, $i \in \text{str-correct}(E)$ iff there exists r_0 , such that for all $r \geq r_0$, $G_E^{(r)}$ contains a path between every process in V_{ir} and i .*

Proof.

\Rightarrow Let $i \in \text{str-correct}(E)$. Since $\text{str-correct}(E) \subseteq \infty\text{-part}(E)$, i belongs also to $\infty\text{-part}(E)$. Take r_0 as the first round such that for all $r \geq r_0$: $V_{ir} \subseteq \infty\text{-part}(E)$. r_0 is well defined since the processes not belonging to $\infty\text{-part}(E)$ can appear only finitely often in the snapshots of i .

Since i is strongly correct, for all $r \geq r_0$, $G_E^{(r)}$ contains a path between every process $j \in \infty\text{-part}(E)$ and i . But as $r \geq r_0$, $V_{ir} \subseteq \infty\text{-part}(E)$. Hence, $G_E^{(r)}$ contains a path between every process of V_{ir} and i .

\Leftarrow Let i be a process and r_0 a round such that for all $r \geq r_0$, $G_E^{(r)}$ contains a path between every process in V_{ir} and i . We need to prove $i \in \text{str-correct}(E)$.

Note that the containment property of IS snapshots guarantees that every process $j \in \infty\text{-part}(E) \setminus V_{ir}$ obtains a snapshot that contains V_{ir} . That is, $(j, i) \in G_E^r$ and hence $(j, i) \in G_E^{(r)}$.

Thus, we conclude that $G_E^{(r)}$ contains a path between every process in $\infty\text{-part}(E)$ and i .

Since there are infinitely many such rounds r and the number of possible paths is bounded, it follows that G_E^* contains a path between every process in $\infty\text{-part}(E)$ and i . Consequently, $i \in \text{str-correct}(E)$.

□

Model simulations. In this paper we focus on models in which every process in a set $1, \dots, n$ alternates writes with taking snapshots of (iterated or non-iterated) memory, using the result of its latest snapshot (or its input value initially) as the value to write. Notice that the updates do not return any meaningful response, just an indication that the operation is complete. Thus, the evolution of the snapshot of a process i in a run E of such a model is characterized by the sequence $V_{i,1}^E, V_{i,2}^E, \dots$ of the snapshots it takes in E .

By simulation of a run of a model B in another model A , we naturally mean a distributed algorithm that in every run of A outputs at every process a sequence of snapshots so that all these sequences are consistent with some run of B and, moreover, reflect the inputs of A . The latter intuitively filters out any “fake” simulation that produces a run of B that has nothing to do with the original run of A .

Formally, in every run E of A , a simulation $Sim_{A,B}$ outputs, at every simulator $i \in \{1, \dots, n\}$ a (finite or infinite) sequence of snapshot values $U_{i,1}, U_{i,2}, \dots$. There exists a run E' of B such that:

- For all i , $V_{i,1}^{E'}, V_{i,2}^{E'}, \dots$ is exactly $U_{i,1}, U_{i,2}, \dots$;
- for every $i \in correct(E)$ (resp., $str\text{-}correct(E)$ if A is an IIS model), $part(E, i) = part(E', i)$.

For the sake of brevity, we assume that in the simulated algorithm, as its local state, each process i simply maintains a vector storing the number of snapshots collected by every other process it is aware of so far. The process writes the vector as its current state in write operation. Each time a new snapshot is taken, the process updates its vector and simply increments its number of steps in it. Initially, the vector of process i stores 1 in position i and 0 at every other position. The reader can easily convince herself that this simplification does not bring a loss of generality, i.e., provided a simulation for such an algorithm, we can derive a simulation for the full-information algorithm.

4 From AS to IIS: resolving and bringing to the front

The goal of this section is to provide an algorithm $AS \rightarrow IIS$ that simulates an execution of an IIS model where the set of processes that appear strongly correct coincides with the set of correct processes (Algorithm 1).

Overview. For each iteration of the IIS model, the processes use the original IS implementation [3]. To ensure fairness of the simulation, each process tries to advance the process that is currently the most behind.

Recall that the IS construction [3] involves n recursive levels, n down to 1, where at each level ℓ , every process registers its participation and then takes an atomic snapshot. If the size of the snapshot is less than ℓ , then the process recursively proceeds to level $\ell - 1$, otherwise it returns the snapshot as its output in the IS simulation. Since at most n processes start at level n and at least one process (the one that writes the last) drops the simulation at each level, at most ℓ processes can reach level ℓ .

In our $AS \rightarrow IIS$ algorithm, in order to promote the next step of a given process, the simulators use an *agreement protocol* [2, 22] for each level of the IS simulation [3]. More precisely, to simulate the atomic snapshot taken by the process in level ℓ , the simulator takes an atomic snapshot itself to compute the set of other simulated processes that also reached level ℓ . If the cardinality of the set is exactly ℓ , then the simulator proposes 1 to the agreement algorithm. Otherwise, it proposes

0. If the agreement protocol returns 1, then the simulated process completes the IS iteration by outputting the set of ℓ processes in level ℓ . If the agreement protocol returns 0, the process gets down to level $\ell - 1$ in the current IS iteration.

To make sure that the simulation is safe, i.e., the simulators indeed agree on the outcome of the simulated step, we use the recently proposed *Resolver Agreement Protocol (RAP)* [11]. This protocol guarantees agreement (no two processes output different values) and validity (every output value was previously proposed). Moreover, if all proposed values are the same, then the algorithm terminates. This feature is implemented using the *commit-adopt (CA)* algorithm [8]. Otherwise, if two different values are proposed, the agreement algorithm may block. The blocked state can be *resolved* by the simulated process itself: the simulated process writes the value it adopted from CA in a dedicated register so that every correct process would eventually read the value and terminate.

Formally, the RAP exports one operation $propose(v)$, $v \in \{0, 1\}$ that returns a value in $\{0, 1, \perp\}$ and is associated with a unique *resolver* process. The following guarantees are provided: (i) Every returned non- \perp value is a proposed value; (ii) If all processes propose the same input value, then no process returns \perp ; (iii) The resolver never returns \perp ; (iv) No two different non- \perp values are returned.

Operation. Algorithm 1 operates as follows. Every process maintains a shared vector $R[i]$, written by i and read by all, that stores i 's perspective on the current simulation. In particular, the sequence of iterations r and levels ℓ that a process j has passed through, *as witnessed by i* , is stored in $R[i, j]$.

After taking a snapshot S in line 8 of the current simulated state, the simulator i first checks if the simulated process i is *blocked* (line 10). A process p is considered blocked if for every $S[j, p]$ that contains (d, r, ℓ) with $(r, \ell) = round\text{-}level(p, S)$, we have $d = blocked$. If simulated process i is blocked in the simulation, simulator i retrieves the round-level (r, ℓ) at which it is blocked (line 25) and participates in $RAP_{i, r, \ell}$. We assign i to be the *resolver* of each RAP instance $RAP_{i, r, \ell}$, and thus the instance returns a non- \perp which “unblocks” simulated process i .

If simulated process i is not blocked, simulator i checks if some process j has completed a *new* (not considered by i in previous rounds of the simulation) round r_j , such that all processes in V_{jr_j} are aware of round r_j of j (line 13). Every such process j is then *frozen* by i , i.e., j is put on hold and not simulated until simulator j performs a “physical” step (in lines 7 or 21).

In the set of remaining processes, the simulator chooses the “slowest” *non-blocked* and *non-frozen* process (line 23). To make sure that the notion of the slowest process is well-defined, we introduce a total order on the tuples (i, r, ℓ) , $i \in \Pi$, $r \in \mathbb{N}$, $\ell \in \mathbb{N}_n$ as follows. We say that $(i, r_i, \ell_i) < (j, r_j, \ell_j)$ if $(r_i < r_j) \vee ((r_i = r_j) \wedge (\ell_i > \ell_j)) \vee ((r_i = r_j) \wedge (\ell_i = \ell_j) \wedge ((i + r_i) \bmod n < (j + r_j) \bmod n))$. This way *argmin* in line 23 returns a single process, ties are broken by choosing the process associated with the current iteration (the association is done in round-robin).

The slowest process p currently observed (by i) in round-level (r, ℓ) is then simulated using p 's next instance of RAP, $RAP_{p, r, \ell}$, which accepts either 1 (exactly ℓ processes have appeared on round-level (r, ℓ) in S) or 0 (otherwise). If $RAP_{p, r, \ell}$ returns 1, p outputs the set of ℓ processes in (r, ℓ) as its snapshot in round r , denoted V_{pr} , and then p is promoted to round $r + 1$ (lines 29 and 30). If $RAP_{p, r, \ell}$ returns 0, p is promoted to level $\ell - 1$ of the same round r (line 33). Otherwise, if $RAP_{p, r, \ell}$ is blocked, we mark the status of i as *blocked* in (r, ℓ) (line 35).

Correctness intuition. Our algorithm tries to always promote the process that is the “most left behind” process to the front of the simulation, unless a process gets *blocked* or *frozen*.

A process i is blocked if two simulators proposed two different values to some $RAP_{i, r, \ell}$, i.e., one simulator finds exactly ℓ processes in (r, ℓ) and, thus, believes that i should complete round r

by outputting the ℓ processes, and the other found strictly less processes in (r, ℓ) and thus believes that i should go one level down in round r and output a smaller snapshot. A process is frozen if it produced a new snapshot in a round r and all the processes appearing in this snapshot became aware of it. The intuition here is that, according to Proposition 1, strongly correct processes are frozen infinitely often. Therefore, only a correct simulator i may appear as strongly correct in the simulated run: otherwise the corresponding simulated process i would get frozen after i crashes and stay frozen forever (only i can “unfreeze” itself in the simulation).

Intuitively, a process i is blocked because another process appeared at its round-level (r, ℓ) and two simulators disagreed whether the other process was there or not: one simulator finds exactly ℓ processes at the level and the other strictly less processes. The last such process p will now be considered the slowest process in the simulation and, thus, will be chosen to be promoted in line 23 by any other simulator. Note that p cannot be blocked in (r, ℓ) , because every simulator that found p in (r, ℓ) will also find exactly ℓ processes in (r, ℓ) . This is because p is the last process to reach (r, ℓ) . Moreover, p completes iteration r having i in its snapshot: since p completes r in level ℓ reached by i , p sees i in round r in the simulated run. By repeating this reasoning inductively, even though i is blocked, another process p carries this information to the “front” of the simulation, thus making sure that every other simulated process will eventually be aware of round r of i . Process i unblocks itself by completing its own $RAP_{i,r,\ell}$ and thus providing it with a non- \perp output.

Thus, intuitively, a correct process i either gets blocked infinitely often or gets frozen infinitely often. In both cases, i is “seen” infinitely often by other correct processes. Moreover, a faulty process is eventually either (i) gets faulty or frozen forever, or (ii) becomes invisible to the remaining processes in the simulated run. In both cases, the faulty process does not appear strongly correct in the simulation. Thus:

Theorem 2 *Algorithm 1 provides a simulation of the IIS model in the AS model such that, for each run E , the simulated run E' satisfies (1) $\text{correct}(E) = \text{str-correct}(E')$, and (2) $\forall i \in \text{correct}(E): \text{part}(E) = \text{part}(E', i)$.*

Proof. Take any run E of Algorithm 1. Recall that the simulated run E' is defined as a collection of all sets V_{ir} , $i = 1, \dots, n$, $r \in \mathbb{N}$ produced in E . By the correctness of the IS simulation [3] and the use of the RAP agreement protocol [11] for each atomic snapshot taken in the simulation of [3], we conclude that for all r , all sets V_{ir} satisfy containment, self-inclusion and immediacy (defined in Section 3). Notice that by the algorithm, every correct process i produces a snapshot V_{ir} in every iteration r .

Every strongly correct process is correct. Assume for the sake of contradiction that $i \in \text{str-correct}(E')$ but $i \notin \text{live}(E)$. Define r_0 to be the first simulated round of E' such that in all $r \geq r_0$, (i) only processes of $\infty\text{-part}(E')$ are simulated and (ii) V_{ir} contain only strongly correct processes. r_0 is well defined since the processes that appear infinitely often in the snapshots of strongly correct processes are necessarily also strongly correct.

Take a round $r \geq r_0$ where V_{ir} is simulated after the crash of i in E (recall that $i \notin \text{correct}(E)$). Since i is strongly correct, all the processes in $\infty\text{-part}(E')$ (including V_{ir}) will eventually be aware of round r of i . But the fact that the processes in V_{ir} are strongly correct means that the processes of $\infty\text{-part}(E')$ are aware of infinitely many of their rounds. Therefore, every process in $\infty\text{-part}(E')$ eventually knows that the processes in V_{ir} were aware of a round of i . Hence, i will be frozen by all of them. But since it has already crashed in E , it will never be unfrozen and cannot be simulated after r , contradiction. Consequently, $i \in \text{correct}(E)$ and $\text{str-correct}(E') \subseteq \text{correct}(E)$.


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1 Shared:  $R[1], \dots, R[n] := [\perp, \dots, \perp], \dots, [\perp, \dots, \perp]$ ;
2 Shared:  $Counter_1, \dots, Counter_n := 0, \dots, 0$ ;
3 Local:  $countf[1, \dots, n] := [0, \dots, 0]$ ; // counters for “frozen” processes
4 Local:  $lastf[1, \dots, n] := [0, \dots, 0]$ ; // last rounds in which processes were “frozen”

5  $R[i, i] := (runm0, n)$ ; // start with highest level of the first iteration
6 while true do
7    $Counter_i ++$ ;
8    $S := \text{snapshot of } R[1], \dots, R[n]$ ;
9   if  $i$  is blocked in  $S$  then
10     $p := i$ ;
11  else
12    for each  $j \in \Pi$  do
13       $x := \text{the largest round such that } V_{jx} \text{ are aware of round } x \text{ of } j \text{ (in } S)$ ;
14      if  $x > lastf[j]$  then
15         $lastf[j] := x$ ;
16         $countf[j] := Counter_j$ ; // freeze  $j$ 
17      end
18    end
19    repeat
20       $cands := \{j \mid j \text{ is not blocked and } Counter_j > countf[j]\}$ ; // ignoring non-participants
21       $Counter_i ++$ ;
22    until  $cands \neq \emptyset$ ;
23     $p := \text{argmin}_{j \in cands} (\text{round-level}(j, S), (j + \text{round}(j, S)) \bmod n)$ ; // choose the “most-behind” process
24  end
25   $(r, \ell) := \text{round-level}(p, S)$ ; // compute current round and level of  $p$ 
26   $U := \{j \mid (*, r, \ell) \in S[*], j\}$ ; // all processes reached  $(r, \ell)$ 
27   $v := RAP_{p, r, \ell}(|U| = \ell)$ ; // the result of next step of  $p$ 
28  if  $v = \text{true}$  then
29     $R[i, p] := S[i, p] \cdot (run, r + 1, n)$ ; //  $p$  completes round  $r$ 
30     $V_{pr} := U$ ; // output the snapshot of  $p$  in round  $r$ 
31  else
32    if  $v = \text{false}$  then
33       $R[i, p] := S[i, p] \cdot (run, r, \ell - 1)$ ; //  $p$  proceeds to  $(r, \ell - 1)$ 
34    else
35       $R[i, p] := S[i, p] \cdot (\text{blocked}, r, \ell)$ ; //  $p$  blocks in  $(r, \ell)$ 
36    end
37  end
38 end

```

Algorithm 1: The $AS \rightarrow IIS$ simulation algorithm: code for process i .

Every correct process is strongly correct. By Proposition 1, there exists a round r_0 such that for all $r \geq r_0$, the processes of V_{ir} are aware of round r of i iff i is strongly correct. Hence, if a process is not strongly correct, the condition of line 13) can apply to it only a finite number of times. Thus, there exists a round r'_0 such that every process that is frozen after it reaches r'_0 is necessarily strongly correct.

Now we show that $correct(E) \subseteq str\text{-}correct(E')$. Suppose not, i.e., there are processes $i, j \in correct(E)$ and a round $r \geq r'_0$ such that i is never aware of round r of j in E' . Since $r \geq r'_0$, j cannot be frozen by i . Let r_i be the round of process i at the moment when j completes round r , i.e., outputs V_{rj} (line 30).

Take r' to be the first round greater than r_i , such that $(i + r') \bmod n + 1 = n$, i.e., r' has the lowest priority in round r' . Thus, before i is simulated at some level ℓ' of r' , any other process that is not frozen or blocked must have completed its simulation of round r' or reached level lower than ℓ' .

Let ℓ' be the level at which i obtains its snapshot in r' and let m be some simulator that

simulated $V_{ir'}$.

We observe first that $(r, \ell) < (r', \ell')$: otherwise, i will eventually reach level $\ell'' \geq \ell'$ of round r , find exactly ℓ'' processes (including j) at that level, and output its snapshot V_{ir} such that $j \in V_{ir}$ —a contradiction with the assumption that i is never aware of round r of j .

Consider the time after i reaches (r', ℓ') and before it obtains the snapshot $V_{ir'}$. By the algorithm, the simulator m must choose the slowest non-blocked and non-frozen process to simulate. Suppose that j is never observed blocked by m after i reaches (r', ℓ') . Since j cannot be frozen by m after r'_0 , the algorithm guarantees that eventually, m would bring j to level (r', ℓ') and, thus, simulates a snapshot $V_{ir'}$ such that $j \in V_{ir'}$ —a contradiction.

Now suppose that m observes j as blocked in round r or later. Without loss of generality, suppose that j is observed as blocked by m in round r . (Indeed, if i is never aware of round r of j in, it is never aware of any later round of j .)

We claim that at the moment the first simulator took its snapshot on behalf of j for round r (in line 8), there was another blocked process k reached (r, ℓ) that later was observed as resolved by another simulator. Indeed, the only reason for j to block in $RAP_{j,r,\ell}$ is that there is another simulator proposing a conflicting set of processes that have been observed to reach (r, ℓ) . Moreover, by the algorithm, since the simulators proposed different values to $RAP_{j,r,\ell}$ one of these sets contains exactly ℓ processes and the other contains strictly less. Consider any process in the difference between these two snapshots (the atomic snapshots taken in line 8 are related by containment [1]). Every such process was considered blocked by one of the simulators at the moment it took its snapshot in line 8, otherwise it would appear in all obtained snapshots or would be chosen to be simulated as slower process. For the last such process s to reach level (r, ℓ) , $RAP_{s,r,\ell}$ cannot get blocked, because all simulators will propose exactly ℓ processes that reached (r, ℓ) . Thus, s obtains V_{sr} such that $j \in V_{sr}$ and enters round $r + 1$.

By our assumption, $(r + 1, n) < (r', \ell')$ and i is never aware of round $r + 1$ of s . Therefore, s is not strongly correct and cannot be frozen as $r + 1 \geq r'_0$. Moreover, s does not block in round r , thus m should eventually try simulating s in round $r + 1$. By repeating the argument inductively, we locate a process t that reaches round $r + 2$ and is aware of round $r + 1$ of s .

Eventually, some process that is aware of round r of j will reach (r', ℓ') , and thus will appear in $V_{ir'}$. Therefore, i is aware of round r of j —a contradiction.

Finally, since every process starts the algorithm by registering its participation at level $(0, n)$ (line 5), the set of participating processes in E is automatically the participating set for every correct process in E' . \square

5 From IIS to AS: identical snapshots and helping

We now describe our $IIS \rightarrow AS$ algorithm that, in any run of the IIS model, simulates a run of the AS model in which every process alternates updates with atomic snapshots [1].

As a basis, we take the non-blocking simulation proposed by Borowsky and Gafni [4]. In this algorithm, each process i maintains a local *counter* vector $C_i[1, \dots, n]$ where each $C_i[j]$ stores the number of simulated snapshots of j *as currently witnessed by i* . To simulate a snapshot operation, process i accesses the iterated memories, writing its counter vector C_i , taking a snapshot of counter vectors of other processes, and updating each position $C_i[k]$ with the maximal value of $C_j[k]$ across all counter vectors read in the iteration. In each iteration r of the IIS memory, this is expressed as

```

1  $C_i[1, \dots, n] := [0, \dots, 0]; C_i[i] := 1; r := 0; SI_i := [0, \dots, 0];$ 
2 while true do
3    $r++;$ 
4    $S := WriteRead_r(C_i, SI_i);$ 
5   if  $\exists SI$  such that  $(\forall (C_j, SI_j) \in S : C_j = SI)$  or  $(\exists (C_j, SI) \in S : SI[i] = C_i[i])$  then
6      $SI_i := SI;$ 
7     output  $SI;$  // Output the next atomic snapshot
8      $C_i[i]++;$ 
9   end
10   $C_i := \max(C_1, \dots, C_n);$  // Adopt the maximal counter value for each process  $j$ 
11 end

```

Algorithm 2: The $IIS \rightarrow AS$ simulation algorithm: code for process i .

a single $WriteRead_r(C_i)$ operation the outcome of which satisfies the self-inclusion, containment, and immediacy properties specified in Section 3. If all these vectors are identical, i outputs the vector as the result of its next snapshot operation. Initially and each time a process i completes its next snapshot operation, it simulates an update operation by incrementing $C_i[i]$.

We first observe that the original simulation of the AS model proposed in [4] is, in the worst case, only non-blocking. Indeed, it admits runs in which some strongly correct process is never able to complete its snapshot operation, even though “noticed” infinitely often. Consider, for example, the following IIS run: $[\{1\}\{2, 3\}], [\{3\}, \{1, 2\}], [\{1\}\{2, 3\}], \dots$, i.e., all the three processes are strongly correct and in every iteration, one of the processes in $\{1, 3\}$ only sees itself and, thus, completes its new snapshot. Thus, in every round one of the processes in $\{1, 3\}$ outputs a new snapshot, while the remaining process 2 sees two different vectors and thus does not complete its simulated snapshot. As a result, process 2 never manages to complete its first snapshot in the simulated AS run, even though it is strongly correct!

To fix this issue, we equip the algorithm of [4] with a helping mechanism, similar to the helping mechanism proposed in the atomic snapshot simulation in [1]. In addition to its counter vector, in each iteration of our Algorithm 2, a process also writes the result of its last snapshot: $WriteRead_r(C_i)$ (line 4). Now a process i outputs a new snapshot not only if it sees that everybody agrees on the clock vector, but also if another process produces a snapshot containing i ’s latest counter value.

Theorem 3 *Algorithm 2 provides a simulation of the AS model in the IIS model such that, for each run E in IIS, the simulated run E' satisfies (1) $str\text{-}correct(E) = correct(E')$ and (2) $\forall i \in str\text{-}correct(E): part(E, i) = part(E')$.*

Proof. Consider any run E of Algorithm 2. First we observe that all atomic snapshots of the simulated processes output in E are all related by containment, i.e., for every two snapshot U and U' output in the algorithm in line 7, we have $U \leq U'$ or $U' \leq U$, when the two vectors are compared position-wise. Indeed, for every output snapshot U , there is a round r and a process i , such that all processes that appear in i ’s immediate snapshot in round r have put U as their clock vectors. Since in the algorithm the clock vector C_i is maintained to have the maximal value seen so far for every process j and by the containment property of immediate snapshot, every process that took the immediate snapshot in round r or later will compute a clock vector $U' \geq U$.

Therefore, we order all atomic snapshots output in E based on the containment order, let U_1, U_2, \dots be the resulting sequence (here $U_\ell \leq U_{\ell+1}$ for each $\ell = 1, 2, \dots$). Then for each $\ell = 1, 2, \dots$ and for each process i , $U_{\ell+1}[i] \neq U_\ell[i]$, we add an update operation in which i increments its counter

(initially 1) and writes the result to position i in the memory just before $U_{\ell+1}$. Notice that since a process only increments its counter after it has output a snapshot, $U_{\ell+1}[i] \neq U_{\ell}[i]$ implies that $U_{\ell+1}[i] = U_{\ell}[i] + 1$.

We call the resulting sequence E' and observe that it is a run of the AS model. Indeed, the snapshots taken in E' are related by containment and, by construction, each snapshot returns the latest written value for each process. By construction, E and E' agree on the sequence of snapshots taken by every given process. Moreover, since the clock vector of process i contains the most up-to-date value for every other process and in the first step each process simply writes its initial clock vector in the memory, the set of participating processes as observed by i in E is the same as the set of participating processes observed by i in E' . Thus, E is an AS run, and Algorithm 2 simulates AS in IIS.

Every update of the counter of a strongly correct process i eventually appears in the snapshot of every other strongly correct process. Thus, every simulated snapshot of a strongly correct process eventually completes and $\text{part}(E, i) = \text{part}(E')$. If a process is not strongly correct, it eventually blocks in trying to complete its snapshot. Thus, $\text{str-correct}(E) = \text{correct}(E')$. \square

6 Conclusion

This paper presents two simulation algorithms that, taken together, maintain the equality between the set of *correct* processes in AS and the set of *strongly correct* processes in IIS. This equality enables a strong equivalence relation between AS and IIS *sub-models*: an object is implementable in an adversarial sub-AS model if and only if it is implementable in the corresponding adversarial sub-IIS model. The result holds regardless of whether the object is one-shot, like a distributed task, or long-lived, like a queue or a counter. (Naturally, in IIS, we guarantee liveness of object operations to the strongly correct processes only.) The equivalence presented in this paper motivates the recent topological characterization of task computability in sub-IIS models [12] and suggests further exploration of iterated models that capture, besides adversaries [5], the use of generic tasks like the Möbius task [14] or of a task from the family of 0-1 exclusion [9].

References

- [1] Y. Afek, H. Attiya, D. Dolev, E. Gafni, M. Merritt, and N. Shavit. Atomic snapshots of shared memory. *J. ACM*, 40(4):873–890, 1993.
- [2] E. Borowsky and E. Gafni. Generalized FLP impossibility result for t -resilient asynchronous computations. In *STOC*, pages 91–100. ACM Press, May 1993.
- [3] E. Borowsky and E. Gafni. Immediate atomic snapshots and fast renaming. In *PODC*, pages 41–51, New York, NY, USA, 1993. ACM Press.
- [4] E. Borowsky and E. Gafni. A simple algorithmically reasoned characterization of wait-free computation (extended abstract). In *PODC*, pages 189–198, 1997.
- [5] C. Delporte-Gallet, H. Fauconnier, R. Guerraoui, and A. Tielmann. The disagreement power of an adversary. *Distributed Computing*, 24(3-4):137–147, 2011.
- [6] M. J. Fischer, N. A. Lynch, and M. S. Paterson. Impossibility of distributed consensus with one faulty process. *J. ACM*, 32(2):374–382, Apr. 1985.

- [7] E. Gafni. On the wait-free power of iterated-immediate-snapshots. Unpublished manuscript, <http://www.cs.ucla.edu/~eli/eli/wfiis.ps>, 1998.
- [8] E. Gafni. Round-by-round fault detectors (extended abstract): Unifying synchrony and asynchrony. In *PODC*, pages 143–152, 1998.
- [9] E. Gafni. The 0-1-exclusion families of tasks. In *OPODIS*, pages 246–258, 2008.
- [10] E. Gafni and P. Kuznetsov. Turning adversaries into friends: Simplified, made constructive, and extended. In *OPODIS*, pages 380–394, 2010.
- [11] E. Gafni and P. Kuznetsov. Relating L -Resilience and Wait-Freedom via Hitting Sets. In *ICDCN*, pages 191–202, 2011.
- [12] E. Gafni, P. Kuznetsov, and C. Manolescu. A generalized asynchronous computability theorem. In *PODC*, 2014.
- [13] E. Gafni and S. Rajsbaum. Distributed programming with tasks. In *OPODIS*, pages 205–218, 2010.
- [14] E. Gafni, S. Rajsbaum, and M. Herlihy. Subconsensus tasks: Renaming is weaker than set agreement. In *International Symposium on Distributed Computing*, pages 329–338, 2006.
- [15] M. Herlihy, D. N. Kozlov, and S. Rajsbaum. *Distributed Computing Through Combinatorial Topology*. Morgan Kaufmann, 2014.
- [16] M. Herlihy and S. Rajsbaum. Simulations and reductions for colorless tasks. In *PODC*, pages 253–260, 2012.
- [17] M. Herlihy and S. Rajsbaum. The topology of distributed adversaries. *Distributed Computing*, 26(3):173–192, 2013.
- [18] M. Herlihy and N. Shavit. The asynchronous computability theorem for t -resilient tasks. In *STOC*, pages 111–120, May 1993.
- [19] M. Herlihy and N. Shavit. The topological structure of asynchronous computability. *J. ACM*, 46(2):858–923, 1999.
- [20] D. N. Kozlov. Chromatic subdivision of a simplicial complex. *Homology, Homotopy and Applications*, 14(1):1–13, 2012.
- [21] P. Kuznetsov. Understanding non-uniform failure models. *Bulletin of the EATCS*, 106:53–77, 2012.
- [22] P. Kuznetsov. Universal model simulation: BG and Extended BG as examples. In *SSS*, pages 17–31, 2013.
- [23] N. Linial. Doing the IIS. Unpublished manuscript, 2010.
- [24] S. Rajsbaum. Iterated shared memory models. In *LATIN*, pages 407–416, 2010.
- [25] S. Rajsbaum, M. Raynal, and C. Travers. The iterated restricted immediate snapshot model. In *COCOON*, pages 487–497, 2008.
- [26] M. Raynal and J. Stainer. Increasing the power of the iterated immediate snapshot model with failure detectors. In *SIROCCO*, pages 231–242, 2012.
- [27] M. Raynal and J. Stainer. Synchrony weakened by message adversaries vs asynchrony restricted by failure detectors. In *PODC*, 2013.